Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 276 (2004) 311–333

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Accurate analytical type solutions for the free in-plane vibration of clamped and simply supported rectangular plates

D.J. Gorman*

Department of Mechanical Engineering, 770 King Edward Avenue, University of Ottawa, Ottawa, Ont., Canada K1N 6N5

Received 30 April 2003; accepted 31 July 2003

Abstract

Highly accurate analytical type solutions are obtained by means of the superposition method for the free in-plane vibration frequencies and mode shapes of fully clamped rectangular plates. The modes are separated into three distinct families and each family is handled separately. Excellent agreement is encountered when computed results are compared with results obtained by other methods. Following the obtaining of solutions for the fully clamped plate, it is shown how this work can be exploited to obtain exact Levy type solutions for the free in-plane vibration modes of the plate with simple support along all edges. The pure shear and extensional vibration modes characteristic of the latter plate problem are handled separately. Again, excellent agreement with results obtained by other means is encountered. A limited number of mode shapes for the fully clamped plate are presented.

1. Introduction

While the number of technical papers related to free in-plane vibration of rectangular plates appearing in the literature is extremely small in comparison to those related to plate lateral vibration there is an ongoing interest in the in-plane vibration subject. Particular applications involve the vibration induced in plates as a result of the action of tangential fluid boundary layers. This type of problem is encountered, for example, in the design of ship hulls.

In an earlier publication the present author addressed the problem of conducting a free in-plane vibration analysis of the completely free rectangular plate [1]. Reference was made to a number of related papers appearing in the literature over the past three decades. Of particular interest was the work of Bardell et al. published in 1996 [2].

*Tel.: +1-613-562-5800; fax: +1-613-562-5177.
E-mail address: dgorman@genie.uottawa.ca (D.J. Gorman).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.07.037

In the earlier paper the superposition method was introduced as an analytical technique for obtaining highly accurate frequencies and mode shapes for the completely free plate. It was shown that application of the method was straightforward and convergence was very rapid. Excellent agreement was obtained upon comparing computed results with those obtained in Ref. [2]. In this latter paper a solution was achieved by means of a Rayleigh–Ritz approach, where a rather complicated set of functions were utilized to represent plate in-plane displacements. It is one of the significant advantages of the superposition method that no such functions need be selected.

The function of the present paper is to compliment the earlier work of the author, by presenting solutions for the fully clamped plate and to demonstrate how exact Levy type solutions are obtained for the two families of vibration modes of the rectangular plate with simple support along all edges. It will then become apparent to the reader that the same approach can be exploited to obtain solutions for plates with various combinations of clamped, free, and simple support edge conditions.

2. Mathematical procedure

2.1. The governing differential equations

Both the governing differential equations and expressions for the in-plane stresses were developed in dimensionless form in the earlier publication [1]. Consequently, only the fully developed version will be presented here in the interest of completeness. The non-dimensionalized differential equations are written as

$$a_{11}\frac{\partial^2 U}{\partial \xi^2}+\frac{a_{12}}{\phi}\frac{\partial^2 V}{\partial \xi\,\partial \eta}+\frac{a_{66}}{\phi}\left[\frac{\partial^2 V}{\partial \xi\,\partial \eta}+\frac{1}{\phi}\frac{\partial^2 U}{\partial \eta^2}\right]+\lambda^4 U=0 \tag{1}$$

and

$$a_{66}\left[\frac{\partial^2 V}{\partial \xi^2}+\frac{1}{\phi}\frac{\partial^2 U}{\partial \xi\,\partial \eta}\right]+\frac{a_{12}}{\phi}\frac{\partial^2 U}{\partial \eta\,\partial \xi}+\frac{a_{12}}{\phi^2}\frac{\partial^2 V}{\partial \eta^2}+\lambda^4 V=0, \tag{2}$$

where all quantities introduced here are defined in the list of nomenclature. Note that λ^2, the dimensionless frequency is defined as, $\lambda^2=\omega a\sqrt{\rho(1-v^2)/E}$. Here ω is the circular frequency of plate vibration and ρ equals plate mass density.

Dimensionless plate in-plane normal and shear stresses are defined, respectively, as

$$\sigma_x^*=\frac{\partial U}{\partial \xi}+\frac{v}{\phi}\frac{\partial V}{\partial \eta},\quad \sigma_y^*=v\frac{\partial U}{\partial \xi}+\frac{1}{\phi}\frac{\partial V}{\partial \eta},\quad \text{and}\quad \tau_{xy}^*=\frac{\partial U}{\partial \eta}+\frac{\partial V}{\partial \xi}.$$

These stress–displacement relationships have been employed to express the basic equilibrium equations (1) and (2) in terms of in-plane plate displacements.

2.2. Analysis of fully clamped plate free vibration

It will be appreciated, in view of the uniform boundary conditions (in-plane displacements normal to and parallel to the edges are everywhere zero), that plate displacement will possess a certain degree of symmetry with respect to the plate central axis. We have chosen to follow the practice established earlier and define a vibration mode as being symmetric about a plate central axis if displacement normal to this axis is symmetrically distributed about it [1]. In that case displacement parallel to the axis will be antisymmetrically distributed about it. Conversely, if displacement normal to a central axis is antisymmetrically distributed about it, then displacement parallel to this axis will be symmetrically distributed about it. We define such modes as being antisymmetrically distributed about the axis.

Accordingly, we define symmetric–symmetric free vibration modes as those where displacements have a symmetric distribution with respect to the two central axes of the plate. Antisymmetric–antisymmetric modes are those for which displacements have an antisymmetric distribution with respect to each of the central axes. Symmetric–antisymmetric modes are those for which displacements have a symmetric distribution with respect to the central ξ-axis and an antisymmetric distribution with respect to the η-axis. We can always choose our central axes orientation so that these conditions are fulfilled for symmetric–antisymmetric mode vibration.

Having classified the possible modes of in-plane vibration for the fully clamped plate in this manner it will be seen that we need analyze one-quarter of the plate only, when examining any of the possible three families of free in-plane vibration modes. It will also be found that, in this way, we avoid the common problem of uncovering double eigenvalues when analyzing square plates.

2.2.1. Symmetric–symmetric modes of the fully clamped plate

Attention is focused on the lower right quarter of the fully clamped plate of interest. This quarter plate segment is shown schematically on the left-hand side of Fig. 1. Small pairs of circles adjacent to the edges $\xi = 0$, and $\eta = 0$, indicate that displacements along these edges satisfy the symmetric mode conditions as discussed earlier. The two forced vibration problems depicted to the right of the figure are to be solved and superimposed, one-upon-the-other. Constants appearing in their solutions are to be so constrained that the net boundary conditions satisfied by the superimposed pair are identical to those to be imposed on the quarter plate of interest (the superposition method).

We now focus attention on the first forced vibration problem, or building block. It also has symmetric boundary conditions imposed on the edges lying along its axes. A condition of zero displacement parallel to the other two edges is to be imposed. The edge, $\eta = 1$, is driven by a harmonic distributed normal stress of circular frequency ω.

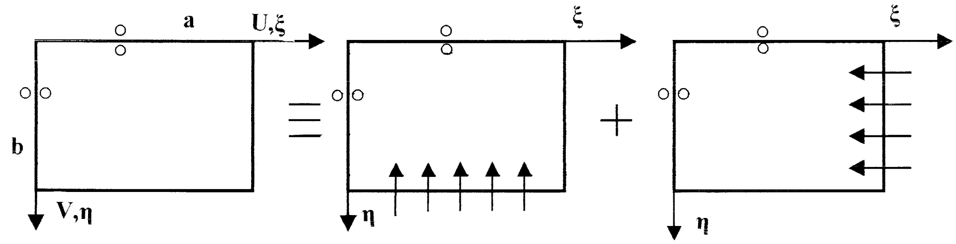

Fig. 1. Building blocks employed in analyzing symmetric–symmetric mode in plane vibration of the fully clamped rectangular plate.

We choose to represent the amplitude of the in-plane displacements of this building block with Levy type series as

$$V(\xi,\eta) = \sum_{m=1,2}^{\infty} V_m(\eta)\sin m\pi\xi \tag{3}$$

and

$$U(\xi,\eta) = \sum_{m=1,2}^{\infty} U_m(\eta)\cos m\pi\xi. \tag{4}$$

It is to be noted that these series satisfy all of the specified boundary conditions along the edges at the extremities of the trigonometric functions as required of Levy type solutions.

Substituting these series representations into the governing differential Eqs. (1) and (2), it is readily shown that, for any value of 'm' we obtain the equations

$$-a_{11}EMPSU_m(\eta) + \frac{a_{12}}{\phi}EMPV'_m(\eta) + \frac{a_{66}}{\phi}\left[EMPV'_m(\eta) - \frac{1}{\phi}U''_m(\eta)\right] + \lambda^4 U_m(\eta) = 0 \tag{5}$$

and

$$-a_{66}\left[EMPS\,V_m(\eta) + \frac{1}{\phi}EMP\,U'_m(\eta)\right] - \frac{a_{12}}{\phi}EMP\,U''_m(\eta) + \frac{a_{12}}{\phi^2}V''_m(\eta) + \lambda^4 V_m(\eta) = 0, \tag{6}$$

where superscripts indicate differentiation with respect to the variable η.

The symbol EMP represents the quantity $m\pi$, with $EMPS$ equals the quantity EMP squared.

In all analytical work pertaining to the quarter plate the quantity λ^2 is non-dimensionalized with respect to the quarter plate edge length 'a'.

Collecting the terms of Eqs. (5) and (6) it is found that we may re-write them as

$$a_{m1}\,U''_m(\eta) + b_{m1}\,V'_m(\eta) + c_{m1}\,U_m(\eta) = 0 \tag{7}$$

and

$$a_{m2}\,V''_m(\eta) + b_{m2}\,U'_m(\eta) + c_{m2}\,V_m(\eta) = 0, \tag{8}$$

where

$$a_{m1} = \frac{a_{66}}{\phi^2},\ \ b_{m1} = EMP(a_{12}+a_{66})/\phi,\ \ c_{m1} = -a_{11}EMPS + \lambda^4,$$

$$a_{m2} = \frac{a_{11}}{\phi^2},\ \ b_{m2} = -\frac{EMP}{\phi}[a_{66}+a_{12}],\ \ \text{and } c_{m2} = \lambda^4 - a_{66}EMPS.$$

Eqs. (7) and (8) constitute a coupled pair of ordinary homogeneous differential equations involving the quantities $U_m(\eta)$ and $V_m(\eta)$.

Through a simple process of differentiating and adding and subtracting of resulting equations, we are able to isolate a single ordinary differential equation involving the quantity $V_m(\eta)$ as follows:

$$V_m^{\mathrm{IV}}(\eta) + b\,V''_m(\eta) + c\,V_m(\eta) = 0, \tag{9}$$

where $b = [a_{m1}c_{m2} - b_{m1}b_{m2} + c_{m1}a_{m2}]/(a_{m1}a_{m2})$, and $c = c_{m1}c_{m2}/(a_{m1}a_{m2})$.

Denoting the square of the roots of the characteristic equation related to Eq. (9) as ε_1^2 and ε_2^2, we have, $\varepsilon_1^2 = (-b + \sqrt{b^2 - 4c})/2$, and $\varepsilon_2^2 = (-b - \sqrt{b^2 - 4c})/2$.

It is found that for all work undertaken here the quantity $b^2 - 4c$ is positive. This means that the quantities ε_1^2 and ε_2^2 will be real, though they may be positive or negative. Denoting those quantities as *Root*1, and *Root*2, respectively, and introducing the quantities, $\beta_m = \sqrt{|Root1|}$, and $\gamma_m = \sqrt{|Root2|}$, it follows that three possible forms of solution for Eq. (9) exist. They are:

Solution 1: Root1 $\geqslant 0.0$, *Root2* $\leqslant 0.0$

$$V_m(\eta) = A_m \sinh \beta_m \eta + B_m \cosh \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta, \tag{10}$$

where A_m, B_m, etc., are constants to be determined.

Solution 2: Root1 $\leqslant 0.0$, *Root2* $\leqslant 0.0$

$$V_m(\eta) = A_m \sin \beta_m \eta + B_m \cos \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta. \tag{11}$$

Solution 3: Root1 $\geqslant 0.0$, *Root2* $\geqslant 0.0$

$$V_m(\eta) = A_m \sinh \beta_m \eta + B_m \cosh \beta_m \eta + C_m \sinh \gamma_m \eta + D_m \cosh \gamma_m \eta. \tag{12}$$

We now turn to obtaining the vibratory response of the first building block of Fig. 1. It will be obvious that all terms antisymmetric about the ξ-axis must be deleted for each of the above solution forms. This is because of the boundary conditions to be imposed along this axis. We focus on the three solution forms, one at a time.

Case 1: Solution 1 applicable

$$V_m(\eta) = B_m \cosh \beta_m \eta + D_m \cos \gamma_m \eta. \tag{13}$$

We next consider the quantity $U_m(\eta)$. Focusing on Eq. (7) it is seen that we may express $U_m(\eta)$ in terms of $U''_m(\eta)$ and $V'_m(\eta)$. However, Eq. (8) permits us to express $U''_m(\eta)$ in terms of $V'''_m(\eta)$ and $V'_m(\eta)$. Utilizing these relationships, along with Eq. (13), above, we obtain

$$U_m(\eta) = B_m \alpha_{2m} \sinh \beta_m \eta + D_m \alpha_{4m} \sin \gamma_m \eta, \tag{14}$$

where $\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$ and $\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}c_{m2})$.

We note that $U_m(\eta)$ has a distribution antisymmetric with respect to the ξ-axis. Next, enforcing the condition that $U_m(\eta)$ equals zero, along the driven edge, we obtain

$$V_m(\eta) = B_m[\cosh \beta_m \eta + \theta_{1m} \cos \gamma_m \eta]$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{15}$$

where $\theta_{1m} = -\alpha_{2m} \sinh \beta_m / (\alpha_{4m} \sin \gamma_m)$.

The distribution of amplitude of the normal harmonic driving stress acting along the edge, $\eta = 1$, is expanded in the same sine series as utilized in Eq. (3), thus,

$$\sigma_y^*|_{\eta=1} = \sum_{m=1,2}^{\infty} E_m \sin m\pi\xi. \tag{16}$$

We may also write for the present building block,

$$\sigma_y^*|_{\eta=1} = \frac{1}{\phi} \frac{\partial V_m(\eta)}{\partial \eta}\bigg|_{\eta=1}. \tag{17}$$

Equating the right-hand sides of Eqs. (20) and (21), we obtain for any value of 'm',

$$V_m(\eta) = E_m\theta_{11m}[\cosh\beta_m\eta + \theta_{1m}\cos\gamma_m\eta] \tag{18}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sinh\beta_m\eta + \alpha_{4m}\theta_{1m}\sin\gamma_m\eta], \tag{19}$$

where $\theta_{11m} = \phi/[\beta_m\sinh\beta_m - \theta_{1m}\gamma_m\sin\gamma_m]$.

We thus have available the response of the building block for any distributed normal driving stress along the edge, $\eta = 1$, when solution 1 is applicable.

It will be appreciated that an identical procedure will be followed to obtain solution for the building block response when the other two forms of solution are applicable. Only the results thus obtained will be provided here.

Case 2: Solution 2 applicable

$$V_m(\eta) = E_m\theta_{11m}[\cos\beta_m\eta + \theta_{1m}\cos\gamma_m\eta] \tag{20}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sin\beta_m\eta + \alpha_{4m}\theta_{1m}\sin\gamma_m\eta], \tag{21}$$

where

$$\theta_{1m} = -\frac{\alpha_{2m}\sin\beta_m}{\alpha_{4m}\sin\gamma_m},\quad \theta_{11m} = -\phi/(\beta_m\sin\beta_m + \theta_{1m}\gamma_m\sin\gamma_m),$$

$$\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}c_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}c_{m2}]/(c_{m1}b_{m2}).$$

Case 3: Solution 3 applicable

$$V_m(\eta) = E_m\theta_{11m}[\cosh\beta_m\eta + \theta_{1m}\cosh\gamma_m\eta] \tag{22}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sinh\beta_m\eta + \theta_{1m}\alpha_{4m}\sinh\gamma_m\eta], \tag{23}$$

where

$$\theta_{1m} = -\frac{\alpha_{2m}\sinh\beta_m}{\alpha_{4m}\sinh\gamma_m},\quad \theta_{11m} = \frac{\phi}{\beta_m\sinh\beta_m + \theta_{1m}\gamma_m\sinh\gamma_m},$$

$$\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}),$$

$$\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

The entire solution for the response of the first building block of Fig. 1 is thus available.

We next focus attention on the second building block of Fig. 1. It has a condition of zero displacement parallel to the edge, $\xi = 1$, to be imposed. This same edge is driven by a distributed harmonic normal stress of circular frequency ω. It will be obvious, therefore, that the solution for this building block can be extracted from that of the first through a proper interchange of variables.

In order to avoid confusion we use the subscript 'n' in connection with the second building block. Also, we will continue to utilize the symbols U and V to designate displacements parallel to the ξ and η axes, respectively. After interchange of the variables ξ and η we express the response of this building block (seeEqs. (3) and (4)) as

$$U(\xi,\eta)=\sum_{n=1,2}^{\infty}U_n(\xi)\sin n\pi\eta \tag{24}$$

and

$$V(\xi,\eta)=\sum_{n=1,2}^{\infty}V_n(\xi)\cos n\pi\eta. \tag{25}$$

Before extracting the solutions for $U_n(\xi)$ and $V_n(\xi)$ from those related to the earlier building block one must proceed as follows:

1. Temporarily replace λ^2 by the quantity $\lambda^2 b/a$.
2. Replace symbols a_{m1}, a_{m2}, etc., of the earlier solution with corresponding symbols a_{n1}, a_{n2}, etc. Also, symbols EMP and $EMPS$ become ENP and $ENPS$, respectively, where $ENP = n\pi$, and $ENPS = ENP$ squared.
3. Temporarily replace the plate aspect ratio with its inverse.

In the case of Solution 1, we will obtain, for example,

$$U_n(\xi)=E_n\theta_{11n}[\cosh\beta_n\xi+\theta_{1n}\cos\gamma_n\xi] \tag{26}$$

and

$$V_n(\xi)=E_n\theta_{11n}[\alpha_{2n}\sinh\beta_n\xi+\theta_{1n}\alpha_{4n}\sin\gamma_n\xi]. \tag{27}$$

The reader will have no difficulty extracting the quantities θ_{1n}, θ_{2n}, α_{2n}, etc., for this solution, and solutions of the other two forms from the material presented for the first building block.

Before beginning to set up the eigenvalue matrix for the symmetric–symmetric mode analysis it is preferable to discuss building block solutions related to the antisymmetric–antisymmetric, and symmetric–antisymmetric families of modes as discussed earlier.

2.2.2. Antisymmetric–antisymmetric modes of the fully clamped plate

The quarter plate and building blocks utilized in connection with the analysis of this mode family differ from those of Fig. 1 in one respect only. Displacements normal to the co-ordinate axes now have a distribution which is antisymmetric with respect to these axes. Displacements parallel to the co-ordinate axes now have a distribution which is symmetric with respect to them.

Levy type solutions for the displacements $V(\xi,\eta)$ and $U(\xi,\eta)$ can now be represented as

$$V(\xi,\eta)=\sum_{m=1,2}^{\infty}V_m(\eta)\cos\frac{(2m-1)\pi\xi}{2} \tag{28}$$

and

$$U(\xi,\eta) = \sum_{m=1,2}^{\infty} U_m(\eta)\sin\frac{(2m-1)\pi\xi}{2}. \tag{29}$$

Substituting the above expressions in the original equilibrium equations we again arrive at the equilibrium Eqs. (7) and (8), where now

$$a_{m1} = \frac{a_{66}}{\phi^2}, \quad b_{m1} = -\frac{EMP}{\phi}[a_{66} + a_{12}], \quad c_{m1} = \lambda^4 - a_{11}EMPS,$$

$$a_{m2} = \frac{a_{11}}{\phi^2}, \quad b_{m2} = \frac{EMP}{\phi}[a_{12} + a_{66}], \text{ and } c_{m2} = \lambda^4 - a_{66}EMPS.$$

Here, $EMP = (2m-1)\pi/2$; and $EMPS = EMP$ squared.

Note that the above quantities, a_{m1}, etc., differ slightly from those used in the symmetric–symmetric mode analysis. In fact, a differential equation governing the quantity $V_m(\eta)$ identical in form to Eq. (9) is now obtained, with the same expressions for quantities b and c as used in the earlier equation. We thus arrive at the same three possible forms of solution for $V_m(\eta)$ as given by Eqs. (10)–(12).

Solutions for the response of the first building block of the present pair are obtained by following a procedure identical to that described for the corresponding building blocks of the earlier pair. The principal difference is that now terms related to the quantity V and symmetric about the ξ-axis must be deleted. Only the results obtained when the analysis is carried out are presented here.

Case 1: Solution 1 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sinh\beta_m\eta + \theta_{1m}\sin\gamma_m\eta], \tag{30}$$

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cosh\beta_m\eta + \theta_{1m}\alpha_{3m}\cos\gamma_m\eta], \tag{31}$$

where

$$\theta_{11m} = \frac{\phi}{[\beta_m\cosh\beta_m + \gamma_m\theta_{1m}\cos\gamma_m]}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cos\gamma_m},$$

$$\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 2: Solution 2 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sin\beta_m\eta + \theta_{1m}\sin\gamma_m\eta], \tag{32}$$

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cos\beta_m\eta + \theta_{1m}\alpha_{3m}\cos\gamma_m\eta], \tag{33}$$

where

$$\theta_{11m} = \frac{\phi}{[\beta_m \cos\beta_m + \theta_{1m}\gamma_m \cos\gamma_m]}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cos\beta_m}{\alpha_{3m}\cos\gamma_m},$$

$$\alpha_{1m} = -\beta_m[a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 3: Solution 3 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sinh\beta_m\eta + \theta_{1m}\sinh\gamma_m\eta], \tag{34}$$

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cosh\beta_m\eta + \theta_{1m}\alpha_{3m}\cosh\gamma_m\eta], \tag{35}$$

where

$$\theta_{11m} = \frac{\phi}{[\beta_m \cosh\beta_m + \gamma_m\theta_{1m}\cosh\gamma_m]}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cosh\gamma_m},$$

$$\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

A solution for the response of the second building block associated with antisymmetric–antisymmetric vibration modes is extracted from the solution given immediately above in a manner identical to that described in connection with the analysis of the symmetric–symmetric mode vibration.

2.2.3. Symmetric–antisymmetric vibration modes of the fully clamped plate

Analysis of this family of free vibration modes is achieved by superposition of the pair of building blocks shown schematically in Fig. 2. Both building blocks have displacement normal to the ξ-axis distributed symmetrically about this axis, and displacement normal to the η-axis distributed antisymmetrically about it.

It will be obvious that Levy type representations for the in-plane displacements U and V of the first building block will again be given by Eqs. (28) and (29) of the antisymmetric–antisymmetric

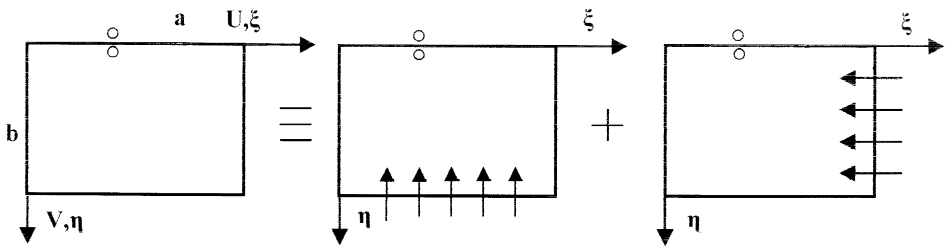

Fig. 2. Building blocks employed in analyzing symmetric–antisymmetric mode in plane vibration of the fully clamped rectangular plate.

mode study. Quantities a_{m1}, b_{m1}, etc., will be identical with those given for the earlier study. The only way the present building block differs from the earlier one lies in the fact that in the expressions for displacement $V_m(\eta)$ of the present building block we must delete those terms antisymmetric about the ξ-axis. This presents no difficulty and the solution for the first building block of the present pair is presented as follows:

Case 1: Solution 1 applicable

$$V_m(\eta) = E_m\theta_{11m}[\cosh\beta_m\eta + \theta_{1m}\cos\gamma_m\eta] \tag{36}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sinh\beta_m\eta + \theta_{1m}\alpha_{4m}\sin\gamma_m\eta], \tag{37}$$

where

$$\theta_{11m} = \frac{\phi}{\beta_m\sinh\beta_m - \theta_{1m}\gamma_m\sin\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{2m}\sinh\beta_m}{\alpha_{4m}\sin\gamma_m},$$

$$\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 2: Solution 2 applicable

$$V_m(\eta) = E_m\theta_{11m}[\cos\beta_m\eta + \theta_{1m}\cos\gamma_m\eta] \tag{38}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sin\beta_m\eta + \theta_{1m}\alpha_{4m}\sin\gamma_m\eta], \tag{39}$$

where

$$\theta_{11m} = \frac{-\phi}{\beta_m\sin\beta_m + \theta_{1m}\gamma_m\sin\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{2m}\sin\beta_m}{\alpha_{4m}\sin\gamma_m},$$

$$\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 3: Solution 3 applicable

$$V_m(\eta) = E_m\theta_{11m}[\cosh\beta_m\eta + \theta_{1m}\cosh\gamma_m\eta] \tag{40}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{2m}\sinh\beta_m\eta + \theta_{1m}\alpha_{4m}\sinh\gamma_m\eta], \tag{41}$$

where

$$\theta_{11m} = \frac{\phi}{\beta_m\sinh\beta_m + \theta_{1m}\gamma_m\sinh\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{2m}\sinh\beta_m}{\alpha_{4m}\sinh\gamma_m},$$

$$\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

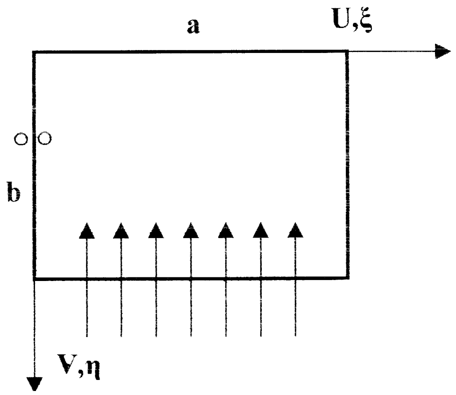

Fig. 3. Intermediate building block utilized to provide solution for second building block of Fig. 2.

and

$$\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

We turn next to the final building block in this pair. It will be obvious that it cannot be extracted through transformation of axes of the solution immediately above. It can, however, be extracted, through interchange of axes, from the building block depicted in Fig. 3. This latter building block differs only slightly from the first building block of the pair employed in analyzing the symmetric–symmetric family of modes (Fig. 1). The single difference lies in the fact that now terms antisymmetric about the ξ-axis must be deleted in expressions for the quantity $V_m(\eta)$.

Accordingly, Levy type solutions for displacements of the building block of Fig. 3 are those given by Eqs. (3) and (4) with expressions a_{m1}, $b_{m1,}$ etc., as given for the earlier solution. Proceeding in the established fashion one obtains as solutions for the quantities $V_m(\eta)$ the following.

Case 1: Solution 1 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sinh\beta_m\eta + \theta_{1m}\sin\gamma_m\eta] \tag{42}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cosh\beta_m\eta + \alpha_{3m}\theta_{1m}\cos\gamma_m\eta], \tag{43}$$

where

$$\theta_{11m} = \frac{\phi}{\beta_m\cosh\beta_m + \theta_{1m}\gamma_m\cos\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cos\gamma_m},$$

$$\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 2: Solution 2 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sin\beta_m\eta + \theta_{1m}\sin\gamma_m\eta] \tag{44}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cos\beta_m\eta + \alpha_{3m}\theta_{1m}\cos\gamma_m\eta], \tag{45}$$

where

$$\theta_{11m} = \frac{\phi}{\beta_m\cos\beta_m + \theta_{1m}\gamma_m\cos\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cos\beta_m}{\alpha_{3m}\cos\gamma_m},$$

$$\alpha_{1m} = -\beta_m[a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

Case 3: Solution 3 applicable

$$V_m(\eta) = E_m\theta_{11m}[\sinh\beta_m\eta + \theta_{1m}\sinh\gamma_m\eta] \tag{46}$$

and

$$U_m(\eta) = E_m\theta_{11m}[\alpha_{1m}\cosh\beta_m\eta + \alpha_{3m}\theta_{1m}\cosh\gamma_m\eta], \tag{47}$$

where

$$\theta_{11m} = \frac{\phi}{\beta_m\cosh\beta_m + \theta_{1m}\gamma_m\cosh\gamma_m}, \quad \theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cosh\gamma_m},$$

$$\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2})$$

and

$$\alpha_{3m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/(c_{m1}b_{m2}).$$

The solution for the response of the second building block of Fig. 2 is, of course, obtained through interchange of axes and by transforming the above solution according to the rules listed earlier. All of the required building block solutions are now available for analysis of the free in-plane vibration of the fully clamped plate.

2.3. Development of eigenvalue matrices

Development of the eigenvalue matrix related to any of the three families of in-plane vibration modes associated with the fully clamped plate is achieved by the following steps completely analogous to those followed earlier in analyzing the rectangular plate free lateral vibration [3] and free in-plane vibration [1]. For illustrative purposes we will describe the generation of the matrix related to the symmetric–symmetric mode analysis, only.

Consider the pair of superimposed building blocks prepared for analysis of this mode family. When superimposed, the solution consisting of the combined pair of building block solutions satisfies exactly the governing differential equations everywhere in the domain of the quarter plate, as well as the prescribed boundary conditions along the co-ordinate axes. We need constrain the unknown driving coefficients only, so that boundary conditions along the other two edges are satisfied.

Let us focus on the edge, $\eta = 1$. The boundary condition to be satisfied here is that the net displacement normal to the edge, V, should equal zero. To achieve this end we expand the contributions of both building blocks toward this displacement in a single appropriate series and require that the net coefficient in each term of this boundary series should vanish. Examining Eq. (3) it is seen that for this problem it is wise to expand the displacements in the sine series of this equation. In this way the contribution of the first building block toward the above displacement is already available in series form. Contributions of the second building block toward this displacement will be available in a series of functions of the variable, ξ (Eq. (25)). These functions are readily expanded in the above sine series following standard Fourier expansion techniques. Upon adding the contributions of each building block to each term in this boundary series, and setting the net coefficient of each term equal to zero we obtain a set of K homogeneous algebraic equations relating the $2K$ driving coefficients $E_m(1), E_m(2), \ldots, E_m(k), E_n(1), E_n(2), \ldots, E_n(K)$, where K equals the number of terms used in the building block solutions as well as the boundary series.

Treating the edge, $\xi = 1$, in an analogous fashion, we finally arrive at a set of $2K$ algebraic equations relating the $2K$ unknown driving coefficients. The eigenvalue matrix for the problem is, in fact, the coefficient matrix of this set of equations. Eigenvalues are those values of the parameter λ^2 which cause the determinant of this matrix to vanish. With the eigenvalue established the associated mode shape of the plate in-plane vibration is obtained by setting one of the non-zero driving coefficients equal to unity, and solving the resulting set of non-homogeneous equations for the remaining coefficients. Eigenvalue matrices related to the other mode families are generated in an identical fashion.

A schematic representation of the eigenvalue matrix generated for the symmetric–symmetric mode analysis is presented in Fig. 4. The small figures above the matrix represent the first and second building blocks employed. Small inserts to the right of the figure indicate the boundary along which a boundary condition is being enforced. Short dashes indicate non-zero matrix elements.

The first row of elements represents the coefficients by which the driving coefficients E_m, E_n, etc., must be multiplied to form the first homogeneous algebraic equation associated with net displacement normal to the edge, $\eta = 1$. The second row represents the second equation, etc. It will be noted that, because of our choice of series to represent net displacements, the upper left quadrant and lower right quadrant of the matrix are of the diagonal type. The lower set of rows in the matrix pertain, of course, to the edge, $\xi = 1$. In the case of a square plate each element in the upper set of rows will equal its corresponding element in the lower set for symmetric–symmetric, and antisymmetric–antisymmetric mode investigations.

Before examining computed results we will look at the analysis of plates with simple support along all edges.

2.4. In-plane free vibration of the simply supported rectangular plate

A rectangular plate is considered to be simply supported if displacements parallel to all edges are forbidden and plate normal stress along each edge equals zero.

It was shown by Lord Rayleigh that exact free vibration frequencies for in-plane vibration of simply supported rectangular plates could be obtained as a special case of the free in-plane

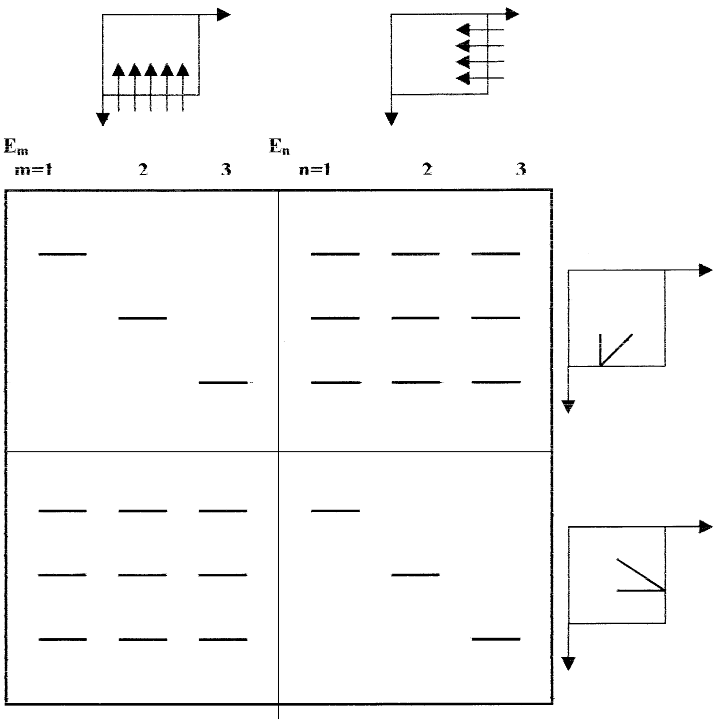

Fig. 4. Schematic representation of eigenvalue matrix utilized in analyzing in-plane vibration modes of fully clamped rectangular plate. Matrix based on three-term building block solutions.

vibration of an infinitely long thin-walled cylindrical shell [4]. The analysis was accomplished by means of energy considerations and the rectangular plate analysis was achieved by letting the shell radius approach infinity. Rayleigh also demonstrated that there were two distinct families of modes associated with simply supported rectangular plate free vibration. In one family, known as 'pure shear modes', the in-plane stresses running normal to the plate edges are everywhere zero. In the other family, known as 'extensional modes', the vibratory motion consists of pure sinusoidal extension in directions perpendicular to the plate edges.

In this paper it is shown that exact Levy type solutions can be obtained for each of the two mode families.

2.4.1. Analysis of the pure shear modes

We begin by returning to the equilibrium Eqs. (1) and (2) and this time express them in terms of in-plane displacements with the condition that $\sigma_x^* = \sigma_y^* = 0$. Proceeding in a manner as described for the general equilibrium equations we arrive at the new set of non-dimensionalized equations:

$$\frac{a_{66}}{\phi^2}\frac{\partial^2 U}{\partial \eta^2} + \frac{a_{66}}{\phi}\frac{\partial^2 V}{\partial \xi\, \partial \eta} + \lambda^4 U = 0 \tag{48}$$

and

$$a_{66}\frac{\partial^2 V}{\partial \xi^2} + \frac{a_{66}}{\phi}\frac{\partial^2 U}{\partial \xi \partial \eta} + \lambda^4 V = 0. \tag{49}$$

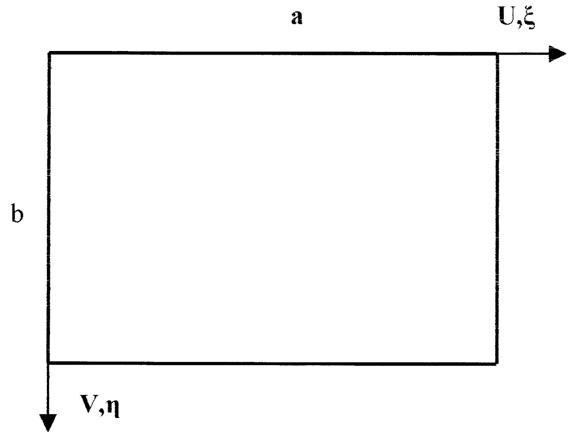

Fig. 5. Co-ordinate system for full rectangular plate with simple support along all edges.

Consider a rectangular plate given simple support, as defined above, along all edges. Let the co-ordinates ξ, and η, run along the upper and left edges of the full plate, respectively, (Fig. 5). We express the displacements U and V as

$$U(\xi,\eta)=\sum_{m=1,2}^{\infty}U_m(\eta)\cos(m-1)\pi\xi \tag{50}$$

and

$$V(\xi,\eta)=\sum_{m=1,2}^{\infty}V_m(\eta)\sin(m-1)\pi\xi. \tag{51}$$

These represent Levy type solutions for displacements of the plate under consideration. For any value of 'm', prescribed boundary conditions along edges at the extremities of the trigonometric functions are completely satisfied. Let us begin by considering solutions associated with any value of m, $m>1$.

Substituting the above forms of solution in the equilibrium equations we obtain

$$a_{m1}U''_m(\eta)+b_{m1}V'_m(\eta)+C_mU_m(\eta)=0 \tag{52}$$

and

$$b_{m2}U'_m(\eta)+C_{m2}V_m(\eta)=0, \tag{53}$$

where

$$a_{m1}=\frac{a_{66}}{\phi^2},\quad b_{m1}=\frac{a_{66}EMP}{\phi},$$
$$c_{m1}=\lambda^4,\quad b_{m2}=\frac{-a_{66}EMP}{\phi}$$

and

$$c_{m2}=-a_{66}\,EMPS+\lambda^4.$$

Here, $EMP=(m-1)\pi$, and $EMPS=EMP$ squared.

Differentiating Eq. (52) once with respect to η and utilizing Eq. (53) to express the quantities $U'''_m(\eta)$ and $U'_m(\eta)$ in terms of $V_m(\eta)$ or its derivatives we arrive at a second order ordinary differential equation governing the quantity $V_m(\eta)$ thus,

$$V''_m(\eta) + \alpha^2 V_m(\eta) = 0, \tag{54}$$

where $\alpha^2 = c_{m1}c_{m2}/(a_{m1}c_{m2} - b_{m1}b_{m2})$.

The quantity α^2 is found to be positive for all problems investigated here. The solution for Eq. (54) is therefore written as

$$V_m(\eta) = A_m \sin \alpha\eta + B_m \cos \alpha\eta, \tag{55}$$

where A_m and B_m are constants to be determined.

$$U_m(\eta) = 0|_{\eta=0};\ \text{this implies } A_m = 0;$$

also

$$U_m(\eta) = 0|_{\eta=1};\ \text{which implies,}$$

$$B_m \sin(\alpha) = 0. \tag{56}$$

With B_m not equal zero Eq. (56) is satisfied only if $\alpha = n\pi$, $n = 1, 2$, etc. The eigenvalues for this problem are then the values of the parameter λ^2 which for any integer values of m and n, $(m \geqslant 2)$, satisfies the equality

$$\alpha^2 = (n\pi)^2. \tag{57}$$

These eigenvalues are easily obtained with a simple computer routine.

We next turn to the situation when m equals 1. Examining Eqs. (50) and (51) it is seen that displacement V will equal zero everywhere while displacement U will be a function of η, only. In this case equilibrium Eq. (48), only, will be applicable and re-arranging it we obtain

$$\frac{\partial^2 U(\eta)}{\partial \eta^2} + \alpha^2 U(\eta) = 0, \tag{58}$$

where here, $\alpha^2 = \lambda^4\phi^2/a_{66}$.

α^2 will always be positive and the solution of Eq. (58) is written as

$$U(\eta) = A \sin \alpha\eta + B \cos \alpha\eta. \tag{59}$$

Since $U(\eta)|_{\eta=0} = 0$, $B = 0$, and since $U(\eta)|_{\eta=1} = 0$.

We write

$$A \sin \alpha = 0. \tag{60}$$

α can therefore only take on values $n\pi$, $n = 1, 2$, etc.

Re-arranging the expression for α^2 we obtain

$$\lambda^2 = n\pi\sqrt{\frac{a_{66}}{\phi^2}}, \tag{61}$$

where n is a positive integer. Exact eigenvalues for this family of modes are therefore obtained immediately from Eq. (61).

2.4.2. Analysis of the extensional modes

We begin by examining the first building block utilized in analyzing the symmetric–symmetric modes of the fully clamped plate. Levy type solutions for the in-plane displacements of this building block are provided by Eqs. (3) and (4). Examining these series solutions it is seen that they satisfy exactly the prescribed simple support boundary conditions along the plate edges, $\xi = 0$, and $\xi = 1$. Expressions for the functions $V_m(\eta)$ and $U_m(\eta)$ as already given by Eqs. (10)–(12), are again applicable. In fact, the same boundary conditions imposed along the edge, $\eta = 0$, for the earlier building block solution are to be imposed here. We must also enforce the condition of zero normal stress along the edge, $\eta = 1$, as well as zero displacement parallel to this edge.

It will be appreciated, in view of the simple support edge conditions, that solution for the quantity $V_m(\eta)$ can be composed of trigonometric terms only. Solution must therefore take the form as given by case 2 associated with the earlier building block, i.e.,

$$V_m(\eta) = B_m[\cos \beta_m \eta + \theta_{1m} \cos \gamma_m \eta] \tag{62}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sin \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{63}$$

where, $\theta_{1m} = -\alpha_{2m} \sin \beta_m / (\alpha_{4m} \sin \gamma_m)$.

Enforcing the boundary condition of zero plate displacement parallel to the edge, $\eta - 1$, we obtain the eigenvalue equation

$$\alpha_{2m} \sin \beta_m + \theta_{1m} \alpha_{4m} \sin \gamma_m = 0. \tag{64}$$

This equation is satisfied for values of $\beta_m = n\pi$, $n = 1, 2$, etc., in which case θ_{1m} will equal zero.

Solution is therefore obtained, with any value of m selected, by searching for values of the parameter λ^2 which cause β_m to take on the above values.

3. Presentation of computed results

3.1. The fully clamped plate

The first step to be taken before computing tabulated theoretical results is to study convergence rates and decide upon a satisfactory value for K, the number of terms to be utilized in the building block solutions. Here we adopt the practice followed earlier in plate lateral vibration studies and compute the eigenvalues to four significant digit accuracy. It is also pointed out that in the interest of conformity with other researchers work, all tabulated eigenvalues presented here are non-dimensionalized with respect to the full plate edge length ‘a’.

In Table 1, eigenvalues are tabulated for a typical convergence test. Computations here are carried to five significant digits for the first symmetric–symmetric mode of the fully clamped square plate, as a function of the parameter K. A value of 0.3 has been used for the Poisson ratio in all calculations reported here.

Table 1
Computed eigenvalues versus K for first symmetric–symmetric mode of the fully clamped square plate

K	λ^2	K	λ^2
3	4.2350	9	4.2350
5	4.2350	11	4.2350
7	4.2350	13	4.2350

Table 2
First four symmetric–symmetric mode eigenvalues, λ^2, for fully clamped plates of aspect ratios 1.0 and 0.5

Mode	$\varphi = 1.0$		$\varphi = 0.5$	
	Present	Ref. [2]	Present	Ref. [2]
1	4.235	4.235	6.712	6.712
2	5.186	5.186	8.140	8.140
3	7.597	—	8.998	—
4	7.800	—	11.57	—

Table 3
First four antisymmetric–antisymmetric mode eigenvalues, λ^2, for fully clamped plates of aspect ratios 1.0 and 0.5

Mode	$\varphi = 1.0$		$\varphi = 0.5$	
	Present	Ref. [2]	Present	Ref. [2]
1	5.859	5.859	7.049	7.049
2	6.708	—	11.25	—
3	7.281	—	11.74	—
4	8.718	—	12.93	—

It is seen in Table 1 that there is no change, even in the fifth significant digit, as parameter K is increased from three to 13. It has been decided to utilize a value of K equal to 11 in all computations for the fully clamped plate. This will provide more than the required convergence as we are interested in four significant digit accuracy in tabulated results.

In Table 2 computed eigenvalues are tabulated, based on the superposition method, for symmetric–symmetric modes of fully clamped plates with aspect ratios of 1.0 and 0.5. A corresponding set of results for antisymmetric–antisymmetric modes is presented in Table 3.

It is found that the most applicable data that can be located for purposes of comparison is that of Bardell et al. [2]. As indicated earlier, they have employed a Rayleigh–Ritz method utilizing a rather complicated set of functions to represent the plate in-plane displacements. Accordingly, where available, their computed results will be tabulated along with the present results for the purposes of comparison.

It will be appreciated that there will be essentially two distinct families of symmetric–antisymmetric modes associated with non-square plates. Both can be studied utilizing the

Table 4
First four symmetric–antisymmetric mode eigenvalues, λ^2, for fully clamped plates of aspect ratios 1.0 and 0.5

Mode	$\varphi = 1.0$		$\varphi = 0.5$		
	Present	Ref. [2]	Present	Ref. [2]	
1	3.555	3.555	4.789	4.789	A–S
2	3.555	3.555	6.379	6.379	S–A
3	5.894	5.895	7.608	—	S–A
4	5.894	5.895	9.515	—	A–S

Table 5
Computed eigenvalues, λ^2, for various mode shapes of the simply supported rectangular plate, $\varphi = 1.0$

m, n	Present	Ref. [2]	m, n	Present	Ref. [2]
0, 1	1.859	1.859	2, 2	5.257	5.257
1, 0	1.859	1.859	0, 3	5.576	5.576
1, 1	2.628	2.628	3, 0	5.576	5.576
0, 2	3.717	3.717	1, 3	5.877	5.877
2, 0	3.717	3.717	3, 1	5.877	5.877
1, 2	4.156	4.156	2, 3	6.701	6.701
2, 1	4.156	4.156	3, 2	6.701	6.701
1, 1*	4.443	4.443	2, 1*	7.025	7.025

*Indicates longitudinal (extensional) vibration. All other modes are of the pure shear type.

symmetric–antisymmetric mode analysis already described. To study modes which are symmetric with respect to the η-axis and antisymmetric with respect to the ξ-axis one simply replaces the aspect ratio of the plate with its inverse. This approach was utilized in preparation of the data of Table 4. Here the symbols to the right of the table, S–A, and A–S, indicate modes related to the plate of aspect ratio 0.5, which are symmetric with respect to the ξ-axis and antisymmetric with respect to the η-axis, or antisymmetric with respect to the ξ-axis and symmetric with respect to the η-axis, respectively.

It will be noted that there is excellent agreement between results computed by means of the present analysis and corresponding results reported by the authors of reference [2]. In almost all cases these results agree to four significant digits. This lends a high degree of confidence with respect to both mathematical procedures. In every case where comparison is possible there is also agreement between mode shapes computed by the present method and those reported in Ref. [2].

3.2. The rectangular plate with simple support along all edges

Following the practice of Bardell et al. [2], we let 'm' and 'n' equal the number of half-waves in displacement running in the ξ and η directions, respectively, for the simply supported rectangular

Table 6
Computed eigenvalues, λ^2, for various mode shapes of the simply supported rectangular plate, $\varphi = 0.5$

m, n	Present	Ref. [2]	m, n	Present	Ref. [2]
1, 0	1.859	1.859	0, 2	7.434	7.434
0, 1	3.717	3.717	4, 0	7.434	7.434
2, 0	3.717	3.717	1, 2	7.663	7.663
1, 1	4.156	4.156	2, 2	8.312	8.312
2, 1	5.257	5.257	4, 1	8.312	8.312
3, 0	5.576	5.576	2, 1*	8.886	8.886
3, 1	6.701	6.701	3, 2	9.293	9.293
1, 1*	7.025	7.025	5, 0	9.293	9.293

*Indicates longitudinal (extensional) vibration. All other modes are of the pure shear type.

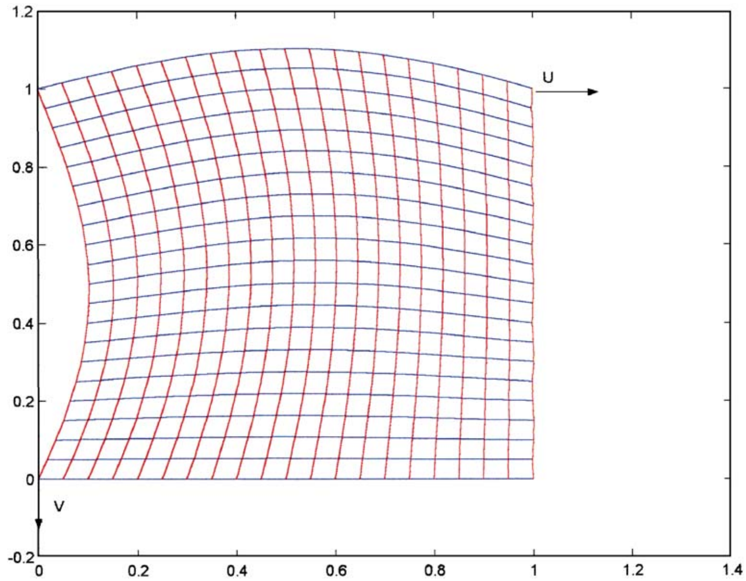

Fig. 6. First symmetric–symmetric mode quarter plate displacements for fully clamped square plate.

plate modes of vibration. Eigenvalues for the various mode shapes of vibration of a square plate as computed by the present analysis are presented in Table 5. Corresponding results reported by Bardell et al. are also presented. for purposes of comparison. They have utilized the energy related analysis of Lord Rayleigh to verify their results for all simply supported plate studies.

In Table 6 results are presented for a simply supported rectangular plate with aspect ratio, b/a, equal to 0.5.

It will be observed that agreement is exact. In the present approach, unlike the Rayleigh–Ritz approach of Ref. [2], no convergence process is involved, and results are exact, as are those of Lord Rayleigh reported in the same reference.

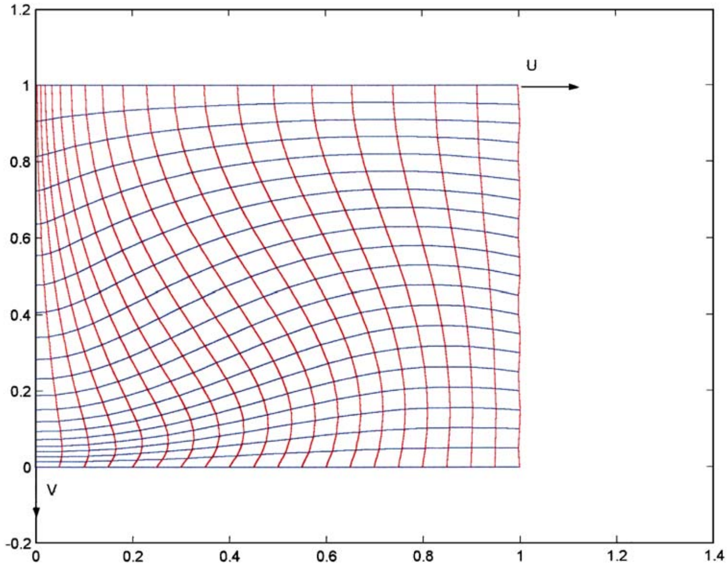

Fig. 7. First antisymmetric–antisymmetric mode quarter plate displacements for fully clamped square plate.

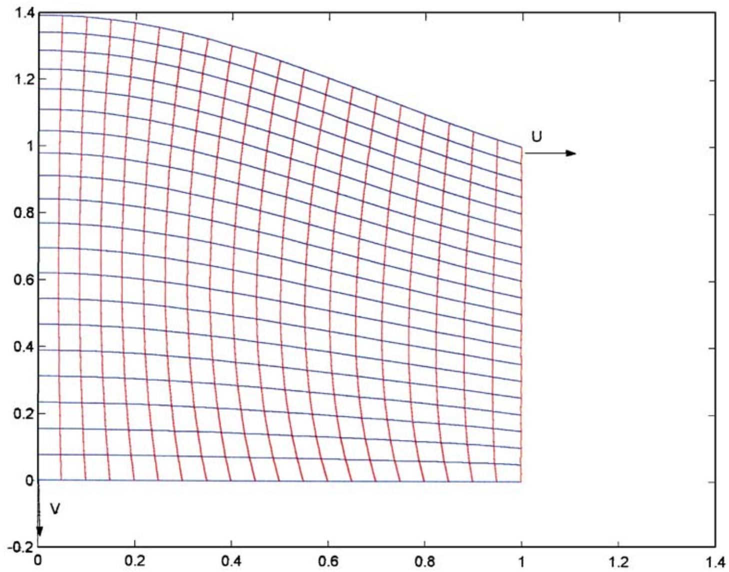

Fig. 8. First symmetric–antisymmetric mode quarter plate displacements for fully clamped square plate.

3.3. Free vibration mode shapes of the fully clamped plate

Computed first mode free in-plane vibration mode shapes for square plates are depicted in Figs. 6–8. These shapes pertain to fully symmetric, fully antisymmetric, and symmetric–antisymmetric mode vibration, respectively. Quarter plate in-plane displacements are depicted, only, since presentation of full plate figures would provide no further information to the reader.

It will be noted that clamped edge conditions (no in-plane motion) are fulfilled along all of the quarter plate outer edges. Symmetric mode conditions are fulfilled along both axes of Fig. 6. Displacements along the axes are everywhere zero. Displacements perpendicular to the axes are non-zero and, in fact, take on extremum values at these quarter plate boundaries. It will be noted, as expected, that this first fully symmetric mode motion is the simplest motion possible, consistent with the imposed boundary conditions.

Similar comments can be made about the first fully antisymmetric mode of Fig. 7. The principal difference is that now the mode shape has an antisymmetric distribution with respect to the axes. It will be noted that there is zero in-plane motion normal to these axes. Here displacements parallel to the axes take on extremum values.

Comments made regarding the above two mode families and their respective boundary conditions apply equally well to the mode shape of Fig. 8. Here, of course, symmetric mode conditions are to be observed along the upper edge of the quarter plate while anti-symmetric mode conditions are observed along the quarter plate left edge. In all of these modes good agreement is encountered on comparing the present results with results inferred from Ref. [2].

Since mode shapes for plates with simple support along all edges involve simple trigonometric functions only, running in both directions, there is nothing to be gained here by depicting this mode shape family.

4. Discussion and conclusions

It is found that the superposition method provides a highly convergent and highly accurate means for computing in-plane free vibration eigenvalues for fully clamped rectangular plates. The governing differential equations are satisfied exactly throughout the domain of the plate and boundary conditions are satisfied to any desired degree of accuracy.

In the case of simply supported plates it is shown that exact free vibration eigenvalues are obtained by means of single term Levy type solutions. Solutions are obtained by utilizing a slightly different approach for the families of pure shear modes and extensional modes. Excellent agreement with results obtained by other mathematical techniques is observed.

It will be apparent to the reader that the superposition method employed here, and in a preceding publication [1], will lend itself readily to the computing of accurate eigenvalues for in-plane vibration of plates with various combinations of classical, free, clamped, and simply supported boundary conditions. It will be obvious that analytical procedures for handling some of these problems are inherent in material presented here. The quarter-plate solution for symmetric–symmetric modes of the fully clamped plate will in fact be identical to that required for analyzing full rectangular plates with two adjacent simply supported and two adjacent clamped edges. The corresponding solution for the completely free plate, [1], handles full rectangular plates with pairs of adjacent simply supported and free edges. This constitutes an additional advantage obtained when it is decided to analyze quarter plate segments where the full plate has uniform boundary conditions.

It is apparent that the present method will constitute a powerful tool for analyzing the in-plane vibration of plates with fixed in-plane support points, attached local masses, in-plane elastic boundary support, etc. Procedures to be followed will be analogous to those utilized in analyzing problems related to plate lateral free vibration. Application of the method to these problems represents a future challenge:

$$V_m(\eta) = A_m \sinh \beta_m \eta + B_m \cosh \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta.$$

Appendix A. Nomenclature

a, b	rectangular plate edge lengths
a_{11}	1.0
a_{12}	v
a_{66}	$(1 - v)/2$
E	Young's modulus of plate material
K	number of terms utilized in building block solutions
u, v	plate in-plane displacements in x and y (ξ and η) directions, respectively
U, V	dimensionless plate displacements, $U = u/a$, $V = v/a$
x, y	rectangular plate coordinates
ξ, η	dimensionless coordinates, $\xi = x/a$, $\eta = y/b$
v	Poisson's ratio (taken as 0.3)
Φ	plate aspect ratio, b/a
λ^2	dimensionless frequency of plate vibration, $\lambda^2 = \omega a\sqrt{\rho(1 - v^2)/E}$
ω	circular frequency of plate vibration
ρ	mass density of plate material
$\sigma_x^*, \sigma_y^*, \tau_{xy}^*$	dimensionless in-plane normal and shear stresses, defined in text

References

[1] D.J. Gorman, Free in-plane vibration analysis of rectangular plates by the method of superposition, *Journal of Sound and Vibration*, in press.

[2] N.S. Bardell, R.S. Langley, J.M. Dunsdon, On the free in-plane vibration of isotropic rectangular plates, *Journal of Sound and Vibration* 191 (3) (1996) 459–467.

[3] D.J. Gorman, *Vibration Analysis of Plates By the Superposition Method*, World Scientific, Singapore, 1999.

[4] Lord Rayleigh, *The Theory of Sound*, Vol. 1, Dover, New York, 1894, pp. 395–407.